

# Superintegrability, isochronicity, and quantum harmonic behavior

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We discuss the properties of superintegrable Hamiltonian systems, in particular those that admit separation of variables in cartesian coordinates. We show that the superintegrability of such potentials is equivalent to the isochronicity of the separated potentials. We use this fact to get a new insight into an old question about the relation between quantum and classical harmonic behavior.

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Hamiltonians of the form

$$H = p_x^2/2 + p_y^2/2 + V_a(x) + V_b(y) \quad (1)$$

admit two independent second-order integrals,  $H_a = p_x^2/2 + V_a(x)$  and  $H_b = p_y^2/2 + V_b(y)$ . They are therefore integrable. Those that admit an additional nontrivial integral of motion are maximally superintegrable since they have  $2n - 1$  integrals of motion,  $n$  being the number of degrees of freedom. Such systems were investigated in [1, 2, 3].

Theorem A relates the existence of this additional integral of motion to the isochronicity of the separated Hamiltonians. It provides a new characterization of isochronous potentials in classical mechanics, which can easily be adapted to many more general contexts. We use this to propose a new definition of quantum harmonic behavior, based on quantum superintegrability. We then give examples that show that potentials that are harmonic according to this definition do not necessarily have equidistant spectra, which was the usual definition for quantum harmonic behavior, but still exhibit regular behavior. The spectra of our examples can indeed be generated from a finite number of states by the application of a creation operator. Therefore their emission spectrum is highly degenerate, which makes them physically very similar to the usual harmonic oscillator.

**Remark 1.** We will assume through this paper that all classical potentials are  $C_2$  over the considered interval of the real axis. The results might be generalizable to  $C_1$ .

**Definition 1.** A Hamiltonian  $H_a(x, p_x)$  and, by extension, the potential  $V_a(x)$  is isochronous if  $V_a$  has a local minimum and if all its bounded trajectories have the same period.

**Remark 2.** Since the potentials considered here are differentiable, and the existence of a local maximum would imply trajectories with infinite period, an isochronous potential can have only one local minimum, and the potential must go to infinity on the boundaries of the considered domain.

**Theorem A.** A Hamiltonian of the form (1) with a local minimum is maximally superintegrable if and only if  $V_a$  and  $V_b$  are isochronous.

*Proof.* The first part of the proof is straightforward. We know, from a theorem due to Nekhoroshev [4], that the bounded trajectories of a maximally superintegrable Hamiltonian are all closed. Since the motion along the  $x$  and  $y$  axis can be decoupled, we can consider separately the periods along each direction. If these periods varied, they would do it continuously with respect to initial conditions that are independent (i.e. the  $x_i(0)$  and  $p_i(0)$ ). We could therefore choose initial conditions such that the ratio between the periods is irrational. In this case the two-dimensional motion would not be periodic, leading to a contradiction.

We now have to demonstrate that the converse is true, i.e. that if  $V_a$  and  $V_b$  are isochronous the whole system is superintegrable. The idea of the proof is to transform each one-dimensional system into a harmonic oscillator, and then use the two-dimensional integrals of the anisotropic harmonic oscillator to find, via the inverse transformation, integrals of the initial system. We will find directly the appropriate transformation using the trajectories of the system.

We can specify a point in the phase space of a one dimensional isochronous potential that has its minimum at  $x = 0$  by specifying the trajectory on which it lies, and the time it took for a particle moving along that trajectory to get there starting from the turning point with  $x > 0$ . Since there is only one such turning point, and since the system is isochronous, this transformation is well defined and one-to-one on the interval  $t \in [0, T)$ ,  $T$  being the period of the system.

We therefore consider the transformation  $(x, p) \rightarrow (r, t)$ , where  $r$  is the value of the turning point with  $x > 0$  and  $t$  the time for a particle to get to  $(x, p)$  from that turning point. We will now introduce a lemma that will be demonstrated below.

**Lemma 1.** The transformation  $(x, p) \rightarrow (r, t)$  is a diffeomorphism for  $r \neq 0$ .

We can compose this change of variables with

$$X = r \cos t, P = r \sin t, \quad (2)$$

which is a transformation from polar to cartesian coordinates and is continuous and differentiable, except at the origin. The transformation  $(x, p) \rightarrow (X, P)$  is therefore a diffeomorphism, except possibly at the origin.

The trajectories in the  $(X, P)$  plane have constant velocity, period  $T$  and are along circles. Therefore  $X$  and  $P$  obey the equations of a harmonic oscillator,

$$\dot{X} = P \quad \dot{P} = -\left(\frac{2\pi}{T}\right)^2 X \quad (3)$$

We use this method for both  $(x_1, p_1)$  and  $(x_2, p_2)$ , and since the periods in the two directions are commensurable, the resulting system expressed in terms of  $\{X_1, X_2, P_1, P_2\}$  exhibits the same motion, and therefore the same integrals as the anisotropic harmonic oscillator with rational ratio. This is known [5] to admit three independent integrals of motion,  $\{Q_1, Q_2, Q_3\}$ .

Since the  $X_i$  and  $P_i$  are differentiable functions of the  $x_i$  and  $p_i$ , The three  $Q_i$ , expressed in terms of the  $\{x_i, p_i\}$  are independent integrals of motion of the initial Hamiltonian.  $\square$

Let us now prove lemma 1.

*Proof.* Differentiability of  $(x, p)$  with respect to time is given by the Hamilton equations,  $\dot{x} = p$  and  $\dot{p} = -V'(x)$ ,

The existence of the two derivatives with respect to  $r$  is given by the theorem of differentiable dependance on initial conditions (see e.g. [6]), applied to the Hamilton equations.

In order to show that  $(x, p) \rightarrow (r, t)$  is also differentiable, we have to show that the determinant of the Jacobian matrix

$$J = \begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial r} \\ \frac{\partial p}{\partial t} & \frac{\partial p}{\partial r} \end{vmatrix} = \begin{vmatrix} p(t, r) & \frac{\partial x}{\partial r} \\ -V'(x(t, r)) & \frac{\partial p}{\partial r} \end{vmatrix}$$

does not vanish on a solution  $(x(t, r), p(t, r))$ . Let us write

$$x(t, r + v) = x_v, \quad p(t, r + v) = p_v, \quad x_0 = x, \quad p_0 = p.$$

The Jacobian reads

$$\begin{aligned} p \frac{\partial p}{\partial r} + V'(x) \frac{\partial x}{\partial r} &= \lim_{v \rightarrow 0} p(p_v - p)/v + V'(x)(x_v - x)/v \\ &= \lim_{v \rightarrow 0} (p_v^2/2 - p^2/2 + V'(x)(x_v - x))/v \\ &\quad - \lim_{v \rightarrow 0} (p_v - p)^2/(2v) \end{aligned}$$

Since  $V'(x)(x_v - x) \simeq V(x_v) - V(x)$  and  $(p_v - p)^2/v \simeq (\partial p/\partial r)^2 v \simeq 0$  we have

$$\begin{aligned} J &= \lim_{v \rightarrow 0} (p_v^2/2 + V(x_v) - (p^2/2 + V(x)))/v \\ &= \lim_{v \rightarrow 0} (V(r) - V(r + v))/v = -V'(r). \end{aligned}$$

This is nonzero everywhere except at  $r = 0$ . The proof of the lemma is complete.  $\square$

We now introduce a concept which will allow us to discuss one-dimensional potentials without always referring to the two-dimensional Hamiltonian.

**Definition 2.** A Hamiltonian  $H_a = p_x^2/2 + V_a(x)$  is called  $2D$ -superintegrable if there exists a  $H_b = p_y^2/2 + V_b(y)$  such that  $H_a + H_b$  is superintegrable. We shall call  $H_b$ , Hamiltonian associated to  $H_a$ . It is also  $2D$ -superintegrable.

**Remark 3.** If  $H_a$  is isochronous, theorem A guarantees that it is  $2D$ -superintegrable. The fact that  $H_a$  is  $2D$ -superintegrable gives no guarantee that it is isochronous, though, unless both  $H_a$  and its associated Hamiltonian have a local minimum.

$2D$ -superintegrability is therefore a generalization of isochronicity since it includes potentials such as free motion and the repulsive harmonic oscillator. It is therefore tempting to identify as "harmonic"  $2D$ -superintegrable potentials in classical mechanics.

Even though some questions regarding the independence of quantum integrals are not yet fully understood [7, 8], the concept of superintegrability can be adapted to quantum mechanics in a straightforward manner, and this had been done since [9]. The adaptation is not so easy for isochronicity. Since well-known examples (such as the harmonic oscillator) have equidistant spectra, it was natural to assume that this property should characterize quantum harmonic potentials [10, 11, 12]. It was shown in [10, 11] that the quantum equivalent to a classical isochronous potential need not be equidistant. We will see that this is still the case when we ask that the quantum equivalent be  $2D$ -superintegrable. The quantum potential will nevertheless exhibit properties that make it similar to the harmonic oscillator, e.g. regarding the infinite degeneracy of the emission spectrum. Therefore it is tempting to identify quantum harmonic behavior and  $2D$ -superintegrability. Studying the properties of  $2D$ -superintegrable potentials will then help us in the search for a more "property-oriented" definition for quantum harmonic behavior.

The formalism of creation-annihilation operators, defined here as operators  $A$  such that  $[H, A] = \lambda A$  with  $\lambda$  a constant, is useful in dealing with  $2D$ -superintegrable potentials. Let us write  $H = H_a(x_1, p_1) + H_b(x_2, p_2)$ , and let  $Q(x_1, x_2, p_1, p_2)$  be the additional integral of motion. If the commutator  $[H_a, Q]$  is nonzero, it is a new integral of  $H$ . This is likely to be a consequence of the definition of the independence of quantum integrals since it is the case in classical mechanics. Therefore the operator  $M : Q \rightarrow [H_a, Q]$  is a nontrivial and nonzero linear operator on the vector space of the integrals of motion of  $H$ . Therefore each nonzero eigenvalue of  $M$  indicates the existence of a creation or annihilation operator for  $H_a$ . Notice that the existence of creation-annihilation operators does not guarantee that the spectrum is equidistant. Consider for example the Hamiltonian  $H = p^2/2 + ax^2 + b/x^2$ , which is classical and quantum  $2D$ -superintegrable for all values of  $a$  and  $b$ . Its spectrum and eigenfunctions are known [3]. In the quantum case we can find a creation-annihilation pair by the method just described, with  $|\lambda| = 2\sqrt{2a\hbar}$ .

For  $b > 3/8\hbar^2$ , the spectrum of this potential is equidistant. For  $-\hbar^2/8 < b < 3\hbar^2/8$ , though, the spectrum is given by

$$E = \sqrt{2a}(2k+1 \pm \nu)\hbar,$$

where  $\nu = \sqrt{1+8b/\hbar^2}/2$ . Therefore, apart from the special case  $b = 0$ , the potentials with  $-1/8\hbar^2 < b < 3/8\hbar^2$ , have a spectrum organized in pairs separated by  $2\sqrt{2a}\hbar$ . Each pair is separated by  $2\sqrt{2a}\hbar$ . The creation-annihilation operators therefore skip a level every time they are applied. Finally notice that  $b$  is proportional to  $\hbar^2$ , and therefore  $\nu$  does not depend on  $\hbar$ . Hence the energy levels are still grouped in pairs as  $\hbar \rightarrow 0$ .

The method of dressing chains (see [13, 14]) can be used to construct Hamiltonians for which we can calculate explicitly the spectrum of energy. The idea is to consider a chain of one-dimensional Hamiltonians  $H_n$  that can be factored as

$$H_n = a_n a_n^\dagger = a_{n+1}^\dagger a_{n+1} + C_n \quad (4)$$

with the periodicity condition  $H_{n+N} = H_n$ . This method is directly related to  $2D$ -superintegrability. Indeed, if one of the Hamiltonians in the chain is  $2D$ -superintegrable, it is possible to construct additional integrals for the whole family of Hamiltonians, for if  $[H_n(x, p_x) + H'(y, p_y), Q] = 0$ , then

$$[H_{n+1}(x, p_x) + H'(y, p_y), a_{n+1}^\dagger Q a_{n+1}] = 0.$$

It turns out indeed that with  $N = 3$ , the first nontrivial case of dressing chains, equations (4) were solved to give a family of  $2D$ -superintegrable potential defined in terms of the fourth Painlevé transcendent (see e.g. [14]), classified as (Q.17) in [2]. Since Hamiltonians satisfying dressing chains have interesting regularity and solvability properties (see [14]), the relation between the existence of dressing chains and integrability deserves further study. Here we will simply use one of the results of [14] to give two further examples of  $2D$ -superintegrable potentials with interesting spectra.

Let us consider the potential

$$V = \frac{\hbar^2 x^2}{8\alpha^4} + \frac{\hbar^2}{(x-\alpha)^2} + \frac{\hbar^2}{(x+\alpha)^2},$$

with  $\alpha \in \mathbb{R}^*$ , which is a special case of the family of potentials classified as (Q.17) in [2].

This potential behaves at infinity like the harmonic oscillator, but has two second-order poles at  $x = \pm\alpha$ . The method of dressing chains allows us in theory to calculate the spectrum for this potential from that of the harmonic oscillator. Indeed, if we write

$$\begin{aligned} b &= \left( p_x - \frac{i\hbar}{2\alpha^2} x - i\hbar f(x) \right) / \sqrt{2} \\ b^\dagger &= \left( p_x + \frac{i\hbar}{2\alpha^2} x + i\hbar f(x) \right) / \sqrt{2} \end{aligned} \quad (5)$$

where  $f(x) = -1/(x-\alpha) - 1/(x+\alpha)$ , we find

$$\begin{aligned} H_1 &= b^\dagger b = \frac{p_x^2}{2} + \frac{\hbar^2 x^2}{8\alpha^4} - \frac{5\hbar^2}{\alpha^2} \\ H_2 &= b b^\dagger = \frac{p_x^2}{2} + \frac{\hbar^2 x^2}{8\alpha^4} + \frac{\hbar^2}{(x-\alpha)^2} + \frac{\hbar^2}{(x+\alpha)^2} - \frac{3\hbar^2}{4\alpha^2}. \end{aligned}$$

For every eigenvector  $|\phi\rangle$  of  $H_1$ , the function  $b|\phi\rangle$  is either zero, or an eigenvector of  $H_2$ . Conversely, if we have an eigenfunction  $|\psi\rangle$  of  $H_2$  the function  $b^\dagger|\psi\rangle$  is zero, or an eigenvector of  $H_1$ . Since the spectrum and eigenvectors of  $H_1$  are well-known, we might get the impression that the problem for  $H_2$  is solved. This is not the case, since the eigenfunctions for  $H_2$  obtained by the application of  $b$  on the analytic, bound states of the harmonic oscillator  $\phi_n(x)$  have a first-order pole at  $x = \pm\alpha$ , and are therefore not square-integrable, unless  $\phi_n(\pm\alpha) = 0$ . Since this is the case only for  $\phi_2$ , and since  $b\phi_2 = 0$ , we can not construct a single square integrable bound state this way.

Let us therefore consider a hypothetical square-integrable ground state  $\psi$  of  $H_2$ . We will now use an important result of Veselov [14] which states that all solutions of  $H_2\psi = E\psi$  (his result applies in fact to any Hamiltonian of the form (Q.17)) must be meromorphic. We can therefore develop  $\psi$  in Laurent series around  $x = \alpha$  or  $x = -\alpha$ . Since  $\psi$  is square integrable and meromorphic, it can not diverge at any finite  $x$ , hence the Laurent series are in fact Taylor series.

Let us now consider the Schrödinger equation  $H_2\psi = E\psi$ . Since  $\psi$  is nonsingular, it must be proportional to  $(x-\alpha)^2(x+\alpha)^2$ . Therefore  $b^\dagger\psi$ , which is an eigenfunction of  $H_1$ , is proportional to  $(x-\alpha)(x+\alpha)$ , and therefore does not diverge at  $x = \pm\alpha$ . The function  $b^\dagger\psi$  is therefore an analytic eigenfunction of the harmonic oscillator. If  $b^\dagger\psi = 0$ , we can solve directly for  $\psi$  and find

$$\psi(x) \propto \frac{\exp \frac{x^2}{4\alpha^2}}{x^2 - \alpha^2}, \quad (6)$$

which is not square integrable. If  $b^\dagger\psi \neq 0$  is square integrable, it is a standard solution of the harmonic oscillator, and can be written as

$$h(x)e^{-\frac{x^2}{4\alpha^2}}$$

where  $h(x)$  is a polynomial in  $x$ .

We have just seen that in this case  $b(b^\dagger\psi) = E\psi$  cannot be square-integrable, and therefore  $E = 0$  and  $b^\dagger\psi = \phi_2$ , the second excited level of the harmonic oscillator.

If we solve  $b^\dagger\psi = \phi_2$ , we find

$$\psi \propto \frac{x(x^2 - \alpha^2)e^{-x^2/4\alpha^2} - \sqrt{2\pi}\alpha^3 e^{x^2/4\alpha^2} \operatorname{erf}\left(\frac{x}{\sqrt{2}\alpha}\right)}{x^2 - \alpha^2}.$$

This function is not square integrable because of its behavior at infinity and at  $x = \pm\alpha$ .

We have therefore proven that if  $\psi$  is a square integrable eigenstate of  $H_2$ , the function  $b^\dagger\psi$  cannot be a square integrable function.

Since it is an analytic eigenstate of the harmonic oscillator, we know from undergraduate textbooks on quantum mechanics (i.e. [15]) that if  $b^\dagger\psi$  is not square integrable, it goes to infinity as a polynomial times  $e^{x^2/4\alpha^2}$ . If that is the case,  $bb^\dagger\psi = E\psi$  cannot be square integrable at infinity unless it is zero. Since we have already found the two states that have  $E = 0$ , we have shown that the Hamiltonian  $H_2$  has no square integrable eigenfunction. This Hamiltonian is therefore equidistant, but in a quite degenerate manner. Interestingly enough, if we consider the same Hamiltonian along the imaginary axis, or equivalently with  $i\alpha \in \mathbb{R}$ , we find a real Hamiltonian without singularities. We can this time find square integrable bound states of  $bb^\dagger$  from states of the harmonic oscillator. We find indeed that the Hamiltonian  $bb^\dagger$  with  $\alpha$  imaginary admits levels with  $E = 0$ , and with  $E = (n+3)\hbar^2/(2\alpha)$  with  $n \in \mathbb{N}$ , but none at  $E = \hbar^2/(2\alpha)$  or  $E = \hbar^2/\alpha$ . We therefore have an example where the creation operator does not skip any level, but where the spectrum is not exactly equidistant either.

The quantum  $2D$ -superintegrable potentials we studied here all have in common with the harmonic oscillator that their spectrum could be generated from a finite number of states by the application of a creation operator. Their emission spectrum is highly degenerate, which is exactly the kind of properties we expect the quantum harmonic potentials to exhibit. Moreover, their classical limit is the (isochronous) harmonic oscillator. Quantum integrals yield classical integrals and it is therefore believed that quantum superintegrable potentials have superintegrable classical equivalents. Therefore quantum  $2D$ -superintegrable Hamiltonians are ex-

pected to have  $2D$ -superintegrable classical equivalents. This provides an additional indication that the use of  $2D$ -superintegrability to generalize isochronicity or harmonicity is appropriate.

All this also provides a new insight on the general significance of superintegrability in quantum mechanics. Superintegrability has already been related to superseparability and exact solvability in quantum mechanics [16]. We have noticed in [2] that quantum  $2D$ -superintegrable potentials known today are all solutions to equations having the Painlevé property. Moreover, the result by Veselov [14], and the relation we established between superintegrability and dressing chains tells us that the eigenstates for these Hamiltonians are often meromorphic in the complex plane. Our paper shows that superintegrability is also related to more physical properties, both in classical and quantum mechanics.

We have shown rigorously the equivalence between  $2D$ -superintegrable potentials with a minimum and isochronous ones. We also showed that this characterization of isochronous potentials could easily be transposed into more general contexts, such as potentials without a local minimum, or, more importantly, to quantum mechanics.  $2D$ -integrability can be easily generalized to Hamiltonians separable in different set of coordinates, or else in higher dimensions. Considering maximally superintegrable three-dimensional potentials of the form  $V_1(x, y) + V_2(z)$ , for example, allows us to deal with  $2D$  isochronous potentials. The results we obtained together with this variety of possible generalizations demonstrates the fruitfulness of the approach presented here.

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